Lecture 2. Schauder estimate I

Notation) $a_{ij}\xi_i\xi_j = \sum_{i,j} a_{ij}\xi_i\xi_j$

$a_{ij}u_{ij} = \sum_{i,j} a_{ij}u_{ij}$

$a_{ij}u_i = \sum_i a_{ij}u_i$

$a_{ij}u_k = a_{ij}u_k$

Thm) Let $\Omega \subset \mathbb{R}^n$ open bounded w/ smooth $\partial\Omega$. Given $\alpha \in (0,1)$

$a_{ij}, b_i, c, f \in C^\alpha(\bar\Omega)$. $g \in C^{2,\alpha}(\bar\Omega)$

$\|a_{ij}\|_{C^\alpha(\bar\Omega)}, \|b_i\|_{C^\alpha}, \|c\|_{C^\alpha} \le \Lambda$

for some $\Lambda \in \mathbb{R}$. In addition, $a_{ij} = a_{ji}$

$a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2$ for some $\lambda > 0$.

Suppose $u \in C^{2,\alpha}(\bar{\Omega})$ satisfies

$$Lu = a_{ij}u_{ij} + b_i u_i + cu = f \quad \text{in } \Omega.$$

$$u = g \quad \text{on } \partial\Omega.$$

Then. $\exists$ $C = C(\alpha, \lambda, \Lambda, n, \Omega)$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C\left(\|f\|_{C^\alpha} + \|g\|_{C^{2,\alpha}} + \sup_{\bar{\Omega}}|u|\right).$$

In particular, C does NOT depend on f and g !!

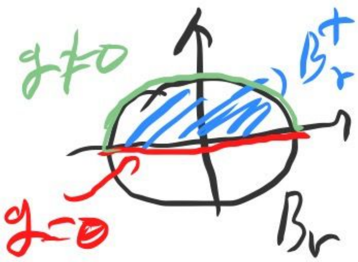
Lemma 1) (Interior Schauder estimate)

$\exists\, C_1 = C_1(n, \alpha, \lambda, \Lambda)$ s.t. if $\Omega = B_2(0)$

then $\|u\|_{C^{2,\alpha}(\overline{B_1(0)})}$

$$\le C_1\left(\|f\|_{C^\alpha(\overline{B_2(0)})} + \sup_{B_2(0)}|u|\right).$$

Notation) $B_r^+(0) = \{\vec{x} \in B_r(0) \mid x_n > 0\}$



$\mathbb{R}^n_+ = \{\vec{x} \in \mathbb{R}^n \mid x_n > 0\}.$

Lemma 2) (Boundary S.E)

$\exists\, C_2 = C_2(n, \alpha, \lambda, \Lambda)$ s.t. if $\Omega = B_2^+(0)$

and $g = 0$ on $\{x_n = 0\}$. then

$$\|u\|_{C^{2,\alpha}(\overline{B_1^+(0)})} \le C_2\left(\|f\|_{C^\alpha(\overline{B_2^+})} + \sup_{B_2^+(0)}|u|\right).$$

$\Longrightarrow$ pset 4 !!

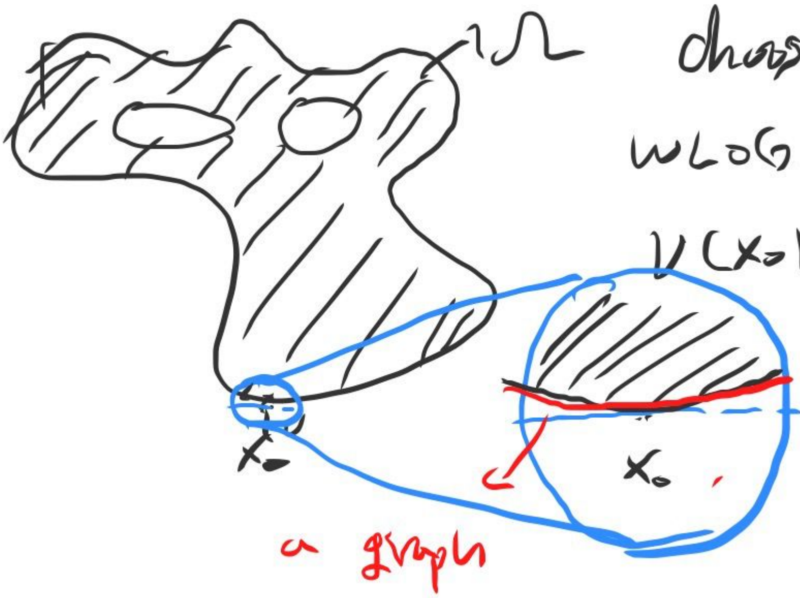
proof of thm).

we consider $v = u - g$

$\Rightarrow v = 0$ on $\partial\Omega$. $Lv = Lu - Lg = f - \underset{C^\alpha}{Lg}$.

It is enough to prove the theorem for the case $g = 0$

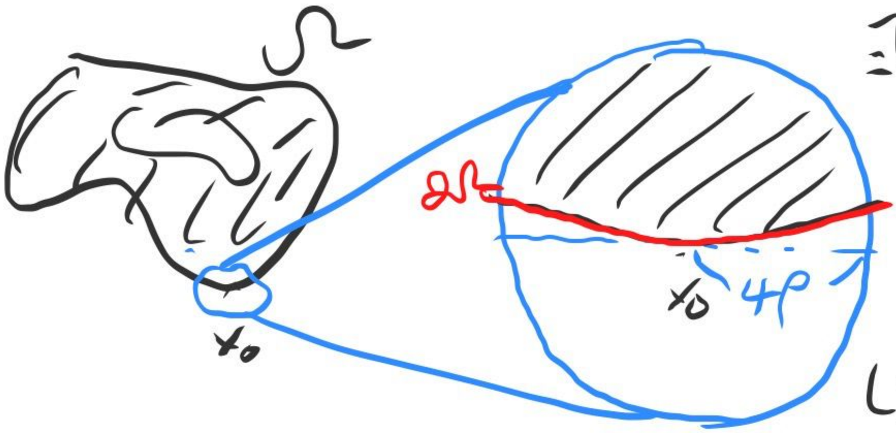
$$
\begin{aligned}
(\because \; \|u\|_{C^{2,\alpha}} &\le \|v\|_{C^{2,\alpha}} + \|g\|_{C^{2,\alpha}} \\
&\le C(\|f\|_{C^\alpha} + \sup|v|) + \|g\|_{C^{2,\alpha}} \\
&\le C(\|f\|_{C^\alpha} + \sup(|u| + |g|)) + \text{"} \\
&\le C(\|f\|_{C^\alpha} + \|g\|_{C^{2,\alpha}} + \sup|u|))
\end{aligned}
$$



choose $x_0 \in \partial\Omega$.

WLOG. we may assume $\nu(x_0) = -\vec{e}_n$.

$\exists$ small $\rho > 0$ only depending on $\partial\Omega$. such that.



$\exists h : B_{4\rho}(0) \to \mathbb{R}$

$B_{4\rho}(0) \subset \mathbb{R}^{n-1}$

such that

$(x', h(x')) \in \partial\Omega$

where $x' = (x_1, \dots, x_{n-1}) \in B_{4\rho}(0)$

By chosing ρ small enough, we have

$h, \nabla h \approx 0$;

Define a map $T : \Omega \cap B_{4\rho}(x_0) \to \mathbb{R}^n_+$ by

$T(x', x_n) = (x, x_n - h(x'))$

Then. $\nabla T \approx I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$., $\|\nabla T - I\|_{C^{0,\alpha}} \approx 0$.,

T is 1-1, onto, T^{-1} is well-defined.!!

we define $\hat{u} : B^+_{2\rho} \to \mathbb{R}$ by $\hat{u}(y) = u(T^{-1}(y))$.

i.e. $u(x) = \hat{u}(T(x))$.

Similarly, we define $\hat{a}_{ij}, \hat{b}_i, \hat{c}, \hat{f}$.

$u(x) = \hat{u}(T(x))$. $\Rightarrow$ $u_i = \hat{u}_k T^k_i$

$u_{ij} = \hat{u}_{kl} T^k_i T^l_j + \hat{u}_k T^k_{ij}$.

where $T = (T^1, \dots, T^n) \in \mathbb{R}^n$

$\Rightarrow \hat{f}(y) = f(T^{-1}y) = a_{ij} u_{ij}(T^{-1}y) + b_i u_i + cu$

$= \hat{a}_{ij} T^k_i T^l_j \hat{u}_{kl} + \hat{a}_{ij} T^k_{ij} \hat{u}_k$

$+ \hat{b}_i T^k_i \hat{u}_k + \hat{c}\hat{u}$.

since $T^k_i \approx \delta_{ki}$. $\Rightarrow$ $\hat{a}_{ij} T^k_i T^l_j$ is u. elliptic

$\hat{a}_{ij} T^k_i T^l_j \xi_k \xi_l \gtrsim a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$

By choosing ρ small enough.

$\hat{a}_{ij} T^k_i T^l_j \xi_k \xi_l \geq \frac{\lambda}{2} |\xi|^2$.

check. $\hat{a}_{ij} T^k_i T^l_j$ is symmetric

$\|\hat{a}_{ij} T^k_i T^l_j\|_{C^\alpha}, \dots, \|\hat{c}\|_{C^\alpha} \leq \Lambda \leq 2\Lambda$.

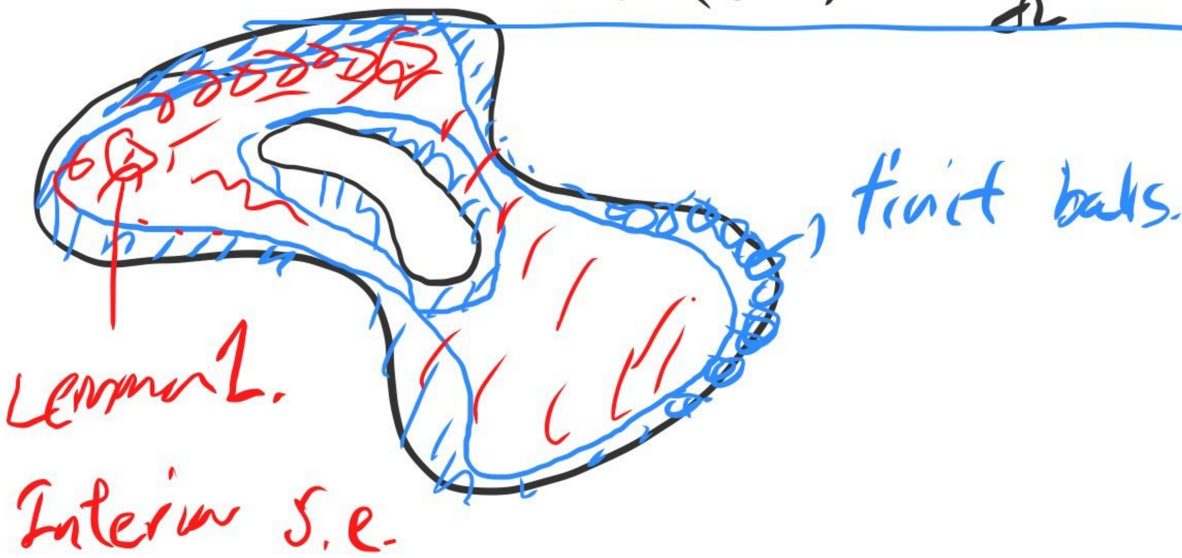
By Lemma 2 (Boundary).

$$\|\hat{u}\|_{C^{2\alpha}} \le C_2 (\|\hat{f}\|_{C^\alpha} + \sup|\hat{u}|)$$

$\Rightarrow$ $\|u\|_{C^{2\alpha}(B_{\rho/2}(x_0))}$

$$\le 2C_2 \left(\|f\|_{C^\alpha(B_{4\rho}(x_0))} + \sup_{B_{4\rho}(x_0)} |u| \right)$$

$$\le 2C_2 \left(\|f\|_{C^\alpha(\Omega)} + \sup_{\Omega} |u| \right)$$



Schauder estimate by "scaling"

by Leon Simon in 1997.

See [GT] section 6 for the classical proof by Schauder.

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what is scaling ??

Given $\lambda > 0$, $b \in \mathbb{R}$, we consider

$$u_\lambda(x) = \lambda^{-b} u(\lambda x).$$

Then. $\nabla u_\lambda = \lambda^{-b+1} \nabla u$

$$\nabla^k u_\lambda = \lambda^{-b+k} \nabla^k u.$$

In particular, $b = k \Rightarrow \nabla^k u_\lambda = \nabla^k u$.

If $b = k+\alpha$, $\Rightarrow [u_\lambda]_{k,\alpha} = [u]_{k,\alpha}$.

$$\|\hat{u}\|_{C^{2,\alpha}(B^+_\rho)} \le C_2 \rho^{-2-\alpha} \left(\|\hat{f}\|_{C^\alpha(B^+_{2\rho})} + \sup|\hat{u}| \right).$$

(for $\rho < 1$)

$\underline{\tilde{u}(x) = \hat{u}(x/\rho)}$, $\quad \hat{u} \in C^{2,\alpha}(B^+_{2\rho})$

$\Rightarrow \tilde{u} \in C^{2,\alpha}(B^+_2)$

$$\Rightarrow \underline{\|\tilde{u}\|_{C^{2,\alpha}(B^+_1)}} \le C_2 \left(\|\tilde{f}\|_{C^\alpha} + \sup|\tilde{u}| \right)$$

$\not\parallel$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hat{u}$

$\|\hat{u}\|_{C^{2,\alpha}}.$

$\nabla \tilde{u} = \rho^{-1} \nabla \hat{u}.$ $\Rightarrow \nabla^2 \tilde{u} = \rho^{-2} \nabla^2 \hat{u}$

$\Rightarrow [\hat{u}]_{2,\alpha} = \rho^{-2-\alpha} [\tilde{u}]_{2,\alpha}.$

Thm 2) $\exists\, C = C(n,\alpha)$, $(\alpha \in (0,1))$
s.t $[D^2 u]_{C^\alpha(\mathbb{R}^n)} \le C [\Delta u]_{C^\alpha(\mathbb{R}^n)}$
holds for all $u \in C^{2,\alpha}(\mathbb{R}^n)$

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proposition) $u : \overline{B_r(0)} \to \mathbb{R}$ is harmonic
Then. $|D^\gamma u(0)| \le \dfrac{C_k}{r^k} \sup\limits_{B_r(0)} |u|$
for $|\gamma| = k$.

where $C_k = C_k(n,k)$

pf of prop) By MVP,

$$u(0) = \fint_{B_r(0)} u \Rightarrow |u(0)| \le \fint |u| \le \sup_{B_r} |u|$$

$\Delta u_i = 0$. By MVP

$$u_i(0) = \fint_{B_r(0)} u_i(x)\,dx = \frac{n}{\omega_n r^n} \int_{B_r(0)} u_i(x)\,dx$$

$$= \frac{n}{\omega_n r^n} \int_{\partial B_r} u(\sigma)\,\nu_i(\sigma)\,d\sigma$$

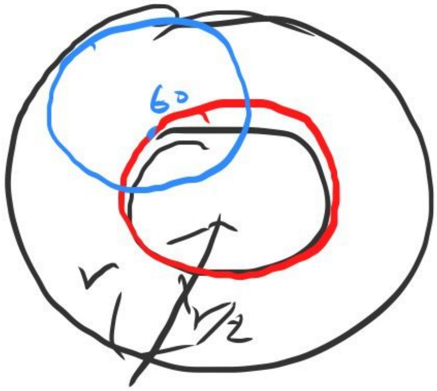
$$|u_i(0)| \le \frac{n}{\omega_n r^n} \int_{\partial B_r} |u(\sigma)|\,d\sigma$$

$$= \frac{n}{r} \fint_{\partial B_r} |u(\sigma)| \le \frac{n}{r} \sup_{B_r(0)} |u|$$

$C_1 = n$.

$\Delta u_{ij} = 0$. $u_{ij}(0) = \fint_{B_{r/2}(0)} u_{ij}$

$$|u_{ij}(0)| \le \frac{2n}{r} \fint_{\partial B_{r/2}} |u_i(\sigma)| d\sigma$$



$$= \frac{2n}{r} |u_i(\sigma_0)|$$

for some $\sigma_0 \in \partial B_{r/2}(0)$

$$|u_i(\sigma_0)| \le \frac{2n}{r} \sup_{B_{r/2}(\sigma_0)} |u| \le \frac{2n}{r} \sup_{B_r} |u|$$

$$|u_{ij}(0)| \le \frac{4n^2}{r^2} \sup_{B_r} |u| \quad !!$$

$C_2 = 4n^2$.

proof of Thm 2)

Towards a contradiction.

we suppose that $\exists$ a seq $u_\ell \in C^{2,\alpha}(\mathbb{R}^n)$

such that $[D^2 u_\ell]_\alpha > \ell [\Delta u_\ell]_\alpha$.

Let $\hat{u}_\ell = \lambda_\ell^{-1} u_\ell$ where $\lambda_\ell = [D^2 u_\ell]_\alpha$.

Then, $[D^2 \hat{u}_\ell]_\alpha = 1$. $[\Delta \hat{u}_\ell]_\alpha < \ell^{-1}$

In particular. $\exists$ $i_\ell, j_\ell, k_\ell \in \{1, \cdots, n\}$

$x_\ell \in \mathbb{R}^n$, $h_\ell > 0$. s.t

$$\frac{|\partial_{i_\ell}\partial_{j_\ell}\hat{u}_\ell(x_\ell + h_\ell e_k) - \partial_{i_\ell}\partial_{j_\ell}\hat{u}_\ell(x_\ell)|}{h_\ell} \geq \frac{1}{2}$$

$$\tilde{u}_\ell = h_\ell^{-2-\alpha}\hat{u}_\ell(x_\ell + h_\ell x)$$

$\Rightarrow [D^2 \tilde{u}_\ell]_\alpha = 1$. $[\Delta \tilde{u}_\ell]_\alpha < \ell^{-1}$

$$|\partial_{i_\ell}\partial_{j_\ell}\tilde{u}_\ell(e_k) - \partial_{i_\ell}\partial_{j_\ell}\tilde{u}_\ell(0)| \geq \frac{1}{2}.$$

$$\check{u}_\ell(x) = \tilde{u}_\ell(x) - \frac{1}{2} x^T \nabla^2 \tilde{u}_\ell(0) x - x^T \nabla \tilde{u}_\ell(0)$$
$$- \tilde{u}_\ell(0)$$

$$\Rightarrow \check{u}_\ell(0) = |\nabla \check{u}_\ell(0)| = |\nabla^2 \check{u}_\ell(0)| = 0$$

$$[D^2 \check{u}_\ell]_\alpha = 1 \, , \quad [\Delta \check{u}_\ell]_\alpha < \ell^{-1}$$

$$|\partial_{i_\ell} \partial_{j_\ell} u_\ell(e_{k_\ell})| \geq 1/2$$

$$\Rightarrow |\check{u}_\ell(x)| \leq C|x|^{2+\alpha}$$

$\Rightarrow \exists$ a subsequence $\check{u}_{\ell_m}$ s.t

$\check{u}_{\ell_m} \longrightarrow u \in C^{2,\alpha}(\mathbb{R}^n)$ by Arzela-Ascoli

$$(i_\ell, j_\ell, k_\ell) \to (\bar{i}, \bar{j}, \bar{k})$$

$$[D^2 u]_\alpha \leq 1, \quad |\partial_{\bar{i}} \partial_{\bar{j}} u(e_{\bar{k}})| \geq 1/2$$

$|u(x)| \leq C|x|^{2+\alpha} = C|x|^{3-(1-\alpha)}$; $\Delta u = 0$!! ※

$0 = u(0) = |Du(0)| = |D^2 u(0)|$, By Lemma $u \equiv 0$ in $\mathbb{R}^n$!!